Prof. Francesco Zumbo

www.francescozumbo.it

Esercizi svolti sugli integrali

ES 1

$$\int \frac{\cos x}{\operatorname{sen}^3 x}\, dx$$

per sostituzione, poniamo $t = \operatorname{sen} x \Rightarrow$

$x = \operatorname{arcsen} t$, $t^3 = \operatorname{sen}^3 x$, $dx = \frac{1}{\sqrt{1-t^2}}\, dt$

dalla relazione fondamentale della goniometria sappiamo

che $\operatorname{sen}^2 x + \cos^2 x = 1$, $\cos^2 x = 1 - \operatorname{sen}^2 x$,

$\cos x = \sqrt{1 - \operatorname{sen}^2 x} \Rightarrow \cos x = \sqrt{1 - t^2}$

adesso abbiamo tutti gli elementi per sostituire

tutti gli elementi della ~~traccia~~

$$\int \frac{\cancel{\sqrt{1-t^2}}}{t^3} \cdot \frac{1}{\cancel{\sqrt{1-t^2}}}\, dt =$$

$$= \int \frac{1}{t^3}\, dt = \int t^{-3}\, dt = \frac{1}{-3+1}\, t^{-3+1} =$$

$$= -\frac{1}{2}\, t^{-2} = -\frac{1}{2}\, \frac{1}{t^2} = -\frac{1}{2t^2} =$$

$$= -\frac{1}{2\,\mathrm{sen}^2 x} + C$$:)

Es 2

$$\int \frac{\ln^3 x}{x}\, dx$$

per sostituzione. Poniamo $t = \ln x$; $e^t = \cancel{e}^{\cancel{\ln} x}$;

$e^t = x$; per cui $dx = e^t \cdot dt$

Sostituendo si ha

$$\int \frac{t^3}{\cancel{e^t}} \cdot \cancel{e^t}\, dt = \int t^3\, dt = \frac{1}{4} t^4 =$$

$$= \frac{1}{4} \ln^4 x + C.$$

$\sim$

Es 3

4

$$\int \frac{\sqrt[3]{\ln x}}{3x}\, dx$$

poniamo $\ln x = t$; $e^{\ln x} = e^{t}$; $x = e^{t}$

$dx = e^{t} \cdot dt$

Sostituendo si ha

$$\int \frac{\sqrt[3]{t}}{3\cancel{e^{t}}} \cdot \cancel{e^{t}} \cdot dt = \int \sqrt[3]{t}\, dt =$$

$$\int t^{\frac{1}{3}}\, dt = \frac{1}{\frac{1}{3}+1} \cdot t^{\frac{1}{3}+1} =$$

$$= \frac{1}{\frac{4}{3}} \cdot t^{\frac{4}{3}} = \frac{3}{4}\sqrt[3]{t^{4}} =$$

$$= \frac{3}{4}\, t\sqrt[3]{t} = \frac{3}{4} \ln x \cdot \sqrt[3]{\ln x} + C$$

Es 4

$$\int \frac{1-\cos x}{(x-\operatorname{sen} x)^2}\, dx$$

5

poniamo $x - \operatorname{sen} x = t$; $dx - \cos x \cdot dx = dt$;

$dx(1-\cos x) = dt$; $dx = \dfrac{dt}{1-\cos x}$

Sostituiamo

$$\int \frac{dt}{t^2} = \int t^{-2}\, dt = \frac{1}{-2+1}\, t^{-2+1} =$$

$$= -t^{-1} = -\frac{1}{t} = -\frac{1}{x-\operatorname{sen} x} + C$$

~

Es 5

⑥

$$\int \frac{x^2}{x^3-9}\,dx$$

poniamo $x^3-9=t$, differenziamo,

$d(x^3-9)\cdot dx = dt$; $3x^2\cdot dx = dt$;

$\frac{x^2\,dx}{3} = \frac{dt}{3}$; $x^2\,dx = \frac{dt}{3}$

$$\int \frac{\frac{dt}{3}}{t} = \int \frac{1}{3}\,dt \cdot \frac{1}{t} =$$

$$= \frac{1}{3}\int \frac{1}{t}\,dt = \frac{1}{3}\ln|t| = \frac{1}{3}\ln|x^3-9| + C$$

Es 6

$$\int \frac{x-1}{1+x^2}\,dx =$$

$$= \int \frac{x}{1+x^2}\,dx - \int \frac{1}{1+x^2}\,dx = \int \frac{2}{2}\,\frac{x}{1+x^2}\,dx - \int \frac{1}{1+x^2}\,dx =$$

$$= \frac{1}{2}\int \frac{2x}{1+x^2}\,dx - \operatorname{arctg} x = \frac{1}{2}\ln|1+x^2| - \operatorname{arctg} x + C$$

~ Es 7

$$\int \frac{\sec^2 x}{tg\, x}\, dx$$

Per definizione $\sec x = \dfrac{1}{\cos x} \implies \sec^2 x = \dfrac{1}{\cos^2 x}$

e sappiamo anche che

$$(tg\, x)' = \frac{1}{\cos^2 x}$$ da cui

$$(tg\, x)' = \sec^2 x \implies$$

$$\int \frac{\sec^2 x}{tg\, x}\, dx = \ln|tg\, x| + C$$

~

~ Es 8

$$\int \operatorname{ctg} x \, dx =$$

$$\int \frac{\cos x}{\operatorname{sen} x} \, dx \qquad \text{ma } (\operatorname{sen} x)' = \cos x$$

per cui

$$\int \frac{\cos x}{\operatorname{sen} x} \, dx = \ln|\operatorname{sen} x| + C$$

~

Es 9

$$\int Th(x)\,dx =$$

$$\int \frac{senh(x)}{cosh(x)}\,dx$$ ma sappiamo che $(cosh\,x)' = senh\,x$

$\Rightarrow$

$$\int \frac{senh(x)}{cosh(x)}\,dx = Ln|cosh(x)| + C$$

Es 10

11

$$\int x^2 \cdot \sqrt[3]{x^3-4}\; dx$$

poniamo $x^3-4=t$; $\sqrt[3]{x^3-4} = \sqrt[3]{t}$;

$x^3 = t+4$; $x = \sqrt[3]{t+4}$; $x = (t+4)^{\frac{1}{3}}$;

$dx = \frac{1}{3}(t+4)^{\frac{1}{3}-1}\, dt$; $dx = \frac{1}{3}(t+4)^{-\frac{2}{3}}\, dt$

$$dx = \frac{1}{3}\,\frac{1}{(t+4)^{\frac{2}{3}}}\, dt \;;$$

$$\boxed{dx = \frac{1}{3}\,\frac{1}{\sqrt[3]{(t+4)^2}}\, dt}$$

Ricaviamo x^2 ; da $x = (t+4)^{\frac{1}{3}}$ $\Rightarrow$

$x^2 = \left[(t+4)^{\frac{1}{3}}\right]^2$;

$$x^2 = (t+4)^{\frac{2}{3}} \quad ; \quad \boxed{x^2 = \sqrt[3]{(t+4)^2}}$$

Procediamo con le sostituzioni

$$\int x^2 \cdot \sqrt[3]{x^3 - 4} \; dx =$$

$$= \int \cancel{\sqrt[3]{(t+4)^2}} \cdot \sqrt[3]{t} \cdot \frac{1}{3} \frac{1}{\cancel{\sqrt[3]{(t+4)^2}}} \; dt =$$

$$= \frac{1}{3} \int \sqrt[3]{t} \; dt = \frac{1}{3} \int t^{\frac{1}{3}} \; dt =$$

$$= \frac{1}{3} \frac{1}{\frac{1}{3} + 1} \cdot t^{\frac{1}{3} + 1} =$$

13

$$= \frac{1}{3} \cdot \frac{1}{\frac{4}{3}} t^{\frac{4}{3}} = \frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{4} \cdot t^{\frac{4}{3}} =$$

$$= \frac{1}{4} t^{\frac{4}{3}} = \frac{1}{4} \sqrt[3]{t^4} = \frac{1}{4} t \sqrt[3]{t} =$$

$$= \frac{1}{4} (x^3 - 4) \cdot \sqrt[3]{x^3 - 4} + C$$

:)

Es 11

$$\int \frac{1 + \operatorname{sen} x}{(x - \cos x)^3}\, dx$$

poniamo $t = x - \cos x$

$dt = dx - (-\operatorname{sen} x)\, dx$; $dt = dx + \operatorname{sen} x\, dx$;

$$\boxed{dt = dx\left(1 + \operatorname{sen} x\right)}$$

$$\int \frac{dt}{t^3} = \int t^{-3}\, dt = \frac{1}{-3+1}\, t^{-3+1} =$$

$$= -\frac{1}{2}\, t^{-2} = -\frac{1}{2}\, \frac{1}{t^2} = -\frac{1}{2(x - \cos x)^2} + C$$

Es 12

$$\int \frac{\arcsin x}{\sqrt{1-x^2}}\, dx$$

poniamo $t = \arcsin x$

$$dt = \frac{dx}{\sqrt{1-x^2}} \quad \Rightarrow$$

$$\int \frac{\arcsin x}{\sqrt{1-x^2}}\, dx = \int t \, dt = \frac{1}{2} t^2 =$$

$$= \frac{1}{2} (\arcsin x)^2 = \frac{1}{2} \arcsin^2 x + C$$

~ Es 13

$$\int \frac{2 \operatorname{arctg} x + 1}{x^2+1}\, dx$$

poniamo $\operatorname{arctg} x = t$; $x = \operatorname{tg} t$;

$$dt = \frac{1}{x^2+1}\, dx$$

per cui l'integrale diviene

$$\int \frac{2t+1}{x^2+1}\, dx = \int 2t+1\, dt = \int 2t\, dt + \int dt =$$

$$= 2\int t\, dt + t = \cancel{2} \cdot \frac{1}{\cancel{2}} t^2 + t = \operatorname{arctg}^2 x + \operatorname{arctg} x =$$

$$= \operatorname{arctg} x \left(\operatorname{arctg} x + 1\right) + C$$

Es 14

$$\int \cos^2 x \cdot \operatorname{sen}^2 x \cdot \cos 2x \; dx$$

noi sappiamo che

$\operatorname{sen}(2x) = 2 \operatorname{sen} x \cdot \cos x$ eleviamo al quadrato

$\operatorname{sen}^2(2x) = 4 \operatorname{sen}^2 x \cdot \cos^2 x$ da cui

$$\frac{\operatorname{sen}^2(2x)}{4} = \operatorname{sen}^2 x \cdot \cos^2 x \quad \Rightarrow$$

$$\operatorname{sen}^2 x \cdot \cos^2 x = \frac{1}{4} \operatorname{sen}^2(2x)$$

L'integrale diviene

$$\int \cos^2 x \cdot \operatorname{sen}^2 x \cdot \cos 2x \, dx = \int \frac{1}{4} \operatorname{sen}^2(2x) \cdot \cos(2x) \, dx =$$

$$= \frac{1}{4} \int \operatorname{sen}^2(2x) \cdot \cos(2x) \, dx =$$

$$= \frac{1}{4} \int \left[\operatorname{sen}(2x) \right]^2 \cdot \cos(2x)\, dx = (2)$$

poniamo $\operatorname{sen}(2x) = t$, differenziamo

$\cos(2x) \cdot 2 \cdot dx = dt$; $dx = \dfrac{dt}{2\cos(2x)}$

Continuiamo ~~cos~~ dalla formula (2)

$$= \frac{1}{4} \int t^2 \cdot \cos(2x)\, dx = \frac{1}{4} \int t^2 \cdot \cancel{\cos(2x)} \cdot \frac{dt}{2\cancel{\cos(2x)}} =$$

$$= \frac{1}{8} \int t^2\, dt = \frac{1}{8} \cdot \frac{1}{3} \cdot t^3 = \frac{1}{24} t^3 =$$

$$= \frac{1}{24} \operatorname{sen}^3(2x) + C$$

Es 15

$$\int x \cdot \ln x \, dx$$

Integriamo per parti

$$\int f(x) \cdot g'(x) \, dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x) \, dx$$

$f(x)$ = fattore finito

$g'(x)$ = fattore differenziale

$f'(x)$ = differenziale del fattore finito

$g(x)$ = integrale del fattore differenziale

poniamo come fattore finito $f(x) = \ln x$

e come fattore differenziale $g'(x) = x$, per cui

$$f'(x) = \frac{1}{x} \quad e \quad g(x) = \int g'(x) \, dx = \int x \, dx = \frac{1}{2} x^2$$

Applichiamo la formula dell'integrazione per parti

$$\int x \cdot \ln x \, dx = \ln x \cdot \frac{1}{2} x^2 - \int \frac{1}{\cancel{x}} \cdot \frac{1}{2} x^{\cancel{2}} \, dx =$$

$$= \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x \, dx = \frac{1}{2} x^2 \cdot \ln x - \frac{1}{2} \cdot \frac{1}{2} x^2 =$$

$$= \frac{1}{2} x^2 \cdot \ln x - \frac{1}{4} x^2 + C = \frac{1}{4} x^2 \left(2 \ln x - x^2 \right) + C$$

~ Es 16 (21)

$$\int x \cdot \text{sen}\, x \, dx$$

poniamo $f(x) = x$ e $g'(x) = \text{sen}\, x$

$f'(x) = 1$ e $g(x) = \int g'(x)\, dx = \int \text{sen}\, x \, dx = -\cos x$

cioè $g(x) = -\cos x$

L'integrale diviene

$$\int x \cdot \text{sen}\, x \, dx = x \cdot (-\cos x) - \int 1 \cdot (-\cos x)\, dx =$$

$$= -x \cos x + \int \cos x \, dx = -x \cdot \cos x + \text{sen}\, x + C$$

~ Es 17

$$\int x \cdot e^x \, dx$$

poniamo $f(x) = x$; $g'(x) = e^x$;

$f'(x) = 1$; $g(x) = e^x$

L'integrale di partenza diviene

$$\int x \cdot e^x \, dx = x \cdot e^x - \int 1 \cdot e^x \, dx =$$

$$= x e^x - e^x + C = e^x (x - 1) + C$$

ES 18

$$\int \ln x \, dx$$

poniamo

$f(x) = \ln x$, $g'(x) = 1$

Da cui

$f'(x) = \frac{1}{x}$; $g(x) = \int g'(x)\, dx = \int 1 \, dx = x$

$g(x) = x$

L'integrale di partenza diviene

$$\int \ln x \, dx = x \cdot \ln x - \int \frac{1}{\cancel{x}} \cdot \cancel{x} \, dx =$$

$$= x \cdot \ln x - \int dx = x \cdot \ln x - x + C =$$

$$= x(\ln x - 1) + C$$

Es 19

$$\int x^2 \cdot \ln x \, dx =$$

$f(x) = \ln x$; $g'(x) = x^2$

Da cui $f'(x) = \frac{1}{x}$ e $g(x) = \int g'(x)\, dx =$

$= \int x^2 \, dx = \frac{1}{3} x^3$, per cui $g(x) = \frac{1}{3} x^3$.

$$\int x^2 \cdot \ln x \, dx = \ln x \cdot \frac{1}{3} x^3 - \int \frac{1}{\cancel{x}} \cdot \frac{1}{3} x^{\cancel{3}\,2} \, dx =$$

$$= \frac{1}{3} x^3 \cdot \ln x - \frac{1}{3} \int x^2 \, dx = \frac{1}{3} x^3 \cdot \ln x -$$

$$- \frac{1}{3} \cdot \frac{1}{3} \cdot x^3 = \frac{1}{3} x^3 \cdot \ln x - \frac{1}{9} x^3 + C =$$

$$= x^3 \left(\frac{1}{3} \ln x - \frac{1}{9} \right) + C$$

ES 20

$$\int x \cdot e^{2x}\, dx$$

poniamo $f(x) = x$ e $g'(x) = e^{2x}$

da cui $f'(x) = 1$ e $g(x) = \int g'(x)\, dx =$

$= \int e^{2x}\, dx$; calcoliamo

$g(x) \int e^{2x}\, dx$; poniamo $e^x = t$, differenziamo

$e^x\, dx = dt$ da cui $dx = \frac{dt}{e^x}$

torniamo all'integrale $g(x)$

$$g(x) = \int e^{2x}\, dx = \int e^{2x} \cdot \frac{dt}{e^x} =$$

$$= \int e^x \cdot \cancel{e^x} \frac{dt}{\cancel{e^x}} = \int e^x\, dt = \int t \cdot dt = \frac{1}{2} t^2 =$$

$$= \frac{1}{2} (e^x)^2 = \frac{1}{2} e^{2x} ;$$

Riassumiamo

$$g(x) = \frac{1}{2} e^{2x}$$

Torniamo all'integrale di partenza

$$\int x \cdot e^{2x} dx = x \cdot \frac{1}{2} e^{2x} - \int 1 \cdot \frac{1}{2} e^{2x} dx =$$

$$= x \cdot \frac{1}{2} e^{2x} - \frac{1}{2} \int e^{2x} d\cancel{x} = (5)$$

ma l'integrale $\int e^{2x} dx$ lo abbiamo già calcolato

poco fa e vale

$$\int e^{2x} dx = \frac{1}{2} e^{2x}$$

Torniamo alla formula (5)

$$= x \cdot \frac{1}{2} e^{2x} - \frac{1}{2} \frac{1}{2} e^{2x} + C = \frac{1}{2} x \cdot e^{2x} -$$

$$- \frac{1}{4} e^{2x} + C = e^{2x} \cdot \left(\frac{1}{2} x - \frac{1}{4} \right) + C$$

:)

ES 21

$$\int \frac{1}{1+e^x}\,dx.$$

poniamo $e^x = t$, differenziamo $e^x \cdot dx = dt$,

$$\int \frac{1}{1+e^x}\,dx = \int \frac{1}{1+t} \cdot \frac{dt}{e^x} =$$

$$\int \frac{1}{1+t} \cdot \frac{dt}{t} = \int \frac{1}{t(1+t)}\,dt =$$

applichiamo la scomposizione per l'integrazione delle funzioni razionali fratte con $N(x)$ di grado 0

$$\frac{1}{t(1+t)} = \frac{A}{t} + \frac{B}{t+1} =$$

$$= \frac{A(t+1)+B\cdot t}{t(t+1)} = \frac{At+A+Bt}{t(t+1)} =$$

$$= \frac{t(A+B)+A}{t(t+1)}$$

Da cui $\begin{cases} A+B=0 \\ A=1 \end{cases}$ $\begin{cases} 1+B=0 \\ A=1 \end{cases}$ $\begin{cases} B=-1 \\ A=1 \end{cases}$

In definitiva si ha

$$\int \frac{1}{t(1+t)}\,dt = \int \frac{1}{t} + \frac{-1}{t+1}\,dt =$$

$$= \int \frac{1}{t}\,dt - \int \frac{1}{t+1}\,dt = \ln|t| - \ln|t+1| + C =$$

$$= \ln \frac{t}{t+1} = \ln \frac{e^x}{e^x+1} + C$$

ES 22

$$\int \sqrt{1-x^2}\; dx$$

poniamo $x = \text{sen}(t)$, $t = \arcsin x$

differenziamo $dx = \cos t \cdot dt$

L'integrale diventa

$$\int \sqrt{1-x^2}\; dx = \int \sqrt{1-\text{sen}^2(t)} \cdot \cos t \cdot dt =$$

$$= \int \cos t \cdot \cos t \cdot dt = \int \cos^2 t \; dt = (1)$$

ma noi sappiamo che per le formule di duplicazione

$$\cos 2t = 2\cos^2 t - 1 \qquad \text{per cui}$$

$$2\cos^2 t = \cos 2t + 1 \; ; \; \cos^2 t = \frac{1+\cos 2t}{2}$$

da (1) si ha

$$\int \cos^2 t \cdot dt = \int \frac{1+\cos 2t}{2}\, dt = \int \frac{1}{2} + \frac{\cos 2t}{2}\, dt =$$

$$= \int \frac{1}{2}\, dt - \int \frac{1}{2} \cos 2t\, dt = \frac{1}{2} t - \frac{1}{2} \int \cos 2t\, dt =$$

$$= \frac{1}{2}\left[t - \int \cos 2t\, dt \right] = (3)$$

Calcoliamo $\int \cos 2t\, dt$

poniamo $2t = u$, $2\,dt = du$; $dt = \frac{du}{2}$

$$\int \cos 2t\, dt = \int \cos u \cdot \frac{du}{2} = \frac{1}{2} \int \cos u\, du = \frac{1}{2} \operatorname{sen} u =$$

$$= \frac{1}{2} \cdot \operatorname{sen}(2t)$$

da (3) si ha

$$= \frac{1}{2}\left[t - \frac{1}{2} \operatorname{sen}(2t) \right] = (4)$$

Ma

$\operatorname{sen} 2t = 2 \operatorname{sen} t \cos t \Rightarrow$

da (4) $= \frac{1}{2}\left[t - \frac{1}{\cancel{2}} \cdot \cancel{2} \operatorname{sen} t \cos t\right] =$

$= \frac{1}{2}\left[t - \operatorname{sen} t \cos t\right] =$ (per la relazione fondamentale della goniometria) $=$

$= \frac{1}{2}\left[t - \operatorname{sen} t \cdot \sqrt{1 - \operatorname{sen}^2 t}\right] =$

$= \frac{1}{2}\left[t - \operatorname{sen} t \cdot \sqrt{1 - \operatorname{sen} t \cdot \operatorname{sen} t}\right] =$

$=$ (essendo $t = \operatorname{arcsen} x$) $=$

$= \frac{1}{2}\left[\operatorname{arcsen} x - \operatorname{sen}(\operatorname{arcsen}(\cancel{x})) \cdot \sqrt{1 - \operatorname{sen}(\operatorname{arcsen} x) \cdot \operatorname{sen}(\operatorname{arcsen} x)}\right] =$

$= \frac{1}{2}\left[\operatorname{arcsen} x - \cancel{x} \cdot \sqrt{1 - x \cdot x}\right] =$

$= \frac{1}{2}\left[\operatorname{arcsen} x - x\sqrt{1 - x^2}\right] + C$:)

Oppure

$$= \frac{1}{2} \operatorname{arcsen} x - \frac{1}{2} x \sqrt{1-x^2} + C$$

ES 23

$$\int \frac{x-8}{\sqrt[3]{x}-2}\,dx$$

Dobbiamo trovare il fattore razionalizzante conveniente

$$\sqrt[3]{x}-\sqrt[3]{2^3} \cdot \frac{\sqrt[3]{x^2}+\sqrt[3]{8x}+\sqrt[3]{64}}{\sqrt[3]{x^2}+\sqrt[3]{8x}+\sqrt[3]{64}}$$

per cui si ha

$$\int \frac{x-8}{\sqrt[3]{x}-2}\,dx = \int \frac{x-8}{\sqrt[3]{x}-2} \cdot$$

$$\cdot \frac{\sqrt[3]{x^2}+\sqrt[3]{8x}+\sqrt[3]{64}}{\sqrt[3]{x^2}+\sqrt[3]{8x}+\sqrt[3]{64}}\,dx$$

$\sqrt[3]{x} = t$; $x^{\frac{1}{3}} = t$;

$\frac{1}{3} x^{\frac{1}{3}-1} \, dx = dt$; $\frac{1}{3} x^{\frac{1-3}{3}} \, dx = dt$;

$\frac{1}{3} x^{-\frac{2}{3}} \, dx = dt$; $\frac{1}{3 x^{\frac{2}{3}}} \, dx = dt$

$dx = 3 \cdot x^{\frac{2}{3}} \cdot dt$

ma se $x^{\frac{1}{3}} = t \Rightarrow x^{\frac{2}{3}} = t^2 \Rightarrow$

$dx = 3t^2 \cdot dt$. Se $x^{\frac{1}{3}} = t \Rightarrow x = t^3$

Torniamo all'integrale

$$\int \frac{x-8}{\sqrt[3]{x} - 2} \, dx = \int \frac{t^3 - 8}{t - 2} \cdot 3t^2 \, dt$$

$$
\begin{array}{rrrr|l}
t^3 & 0t^2 & 0t^1 & -8 & t-2 \\
\cline{5-5}
-t^3 & +2t^2 & & & t^2+2t+4 \\
\hline
/ & 2t^2 & 0t & -8 & \\
 & -2t^2 & +4t & & \\
\cline{2-4}
 & / & 4t & -8 & \\
 & & -4t & +8 & \\
\cline{3-4}
 & & / & / &
\end{array}
$$

$$(t^3-8)=(t-2)(t^2+2t+4)$$

$$\frac{t^3-8}{t-2}=t^2+2t+4$$

Torniamo all'integrale

$$\int (t^2+2t+4)\cdot(3t^2)\ dt =$$

$$= \int 3t^4+6t^3+12t^2\ dt = 3\int t^4\,dt + 6\int t^3\,dt + 12\int t^2\cdot dt =$$

$$= 3\cdot\frac{1}{5}t^5 + \cancel{6}^{\,3}\frac{1}{\cancel{4}_{2}}t^4 + \cancel{12}^{\,4}\cdot\frac{1}{\cancel{3}}t^3 =$$

$$= \frac{3}{5}t^5 + \frac{3}{2}t^4 + 4t^3 = \frac{3}{5}\left(\sqrt[3]{x}\right)^5 + \frac{3}{2}\left(\sqrt[3]{x}\right)^4 +$$

$$+4\left(\sqrt[3]{x}\right)^3 = \frac{3}{5}\sqrt[3]{x^5} + \frac{3}{2}\sqrt[3]{x^4} + 4\sqrt[3]{x^3} =$$

$$= \frac{3}{5}\,x\sqrt[3]{x^2} + \frac{3}{2}\,x\sqrt[3]{x} + 4x \Rightarrow$$

$$\int \frac{x-8}{\sqrt[3]{x}-2}\,dx = \frac{3}{5}x\sqrt[3]{x^2} + \frac{3}{2}x\sqrt[3]{x} + 4x + C$$

ES 24

$$\int \sqrt{\operatorname{sen} x} \cdot \cos^3 x \, dx$$

procediamo per sostituzione $\operatorname{sen} x = t$

differenziamo $\cos x \cdot dx = dt$

$$\int \sqrt{\operatorname{sen} x} \cdot \cos^2 x \cdot \cos x \, dx =$$

$$= \int \sqrt{t} \cdot \cos^2 x \cdot dt =$$

$$= \int \sqrt{t} \cdot (1 - \operatorname{sen}^2 x) \, dt =$$

$$\int \sqrt{t} \cdot (1 - t^2) \, dt = \int \sqrt{t} - \sqrt{t} \cdot t^2 \, dt =$$

$$= \int t^{\frac{1}{2}} - t^{\frac{1}{2}} \cdot t^{2} \, dt =$$

$$= \int t^{\frac{1}{2}} \, dt - \int t^{\frac{1+4}{2}} \, dt = \int t^{\frac{1}{2}} \, dt - \int t^{\frac{5}{2}} \, dt =$$

$$= \frac{1}{\frac{1}{2}+1} t^{\frac{1}{2}+1} - \frac{1}{\frac{5}{2}+1} t^{\frac{5}{2}+1} =$$

$$= \frac{1}{\frac{1+2}{2}} t^{\frac{1+2}{2}} - \frac{1}{\frac{5+2}{2}} \cdot t^{\frac{5+2}{2}} =$$

$$= \frac{1}{\frac{3}{2}} t^{\frac{3}{2}} - \frac{1}{\frac{7}{2}} \cdot t^{\frac{7}{2}} =$$

$$= \frac{2}{3} t^{\frac{3}{2}} - \frac{2}{7} t^{\frac{7}{2}} = \frac{2}{3} \sqrt[2]{t^{3}} - \frac{2}{7} \sqrt[2]{t^{7}} =$$

$$= \frac{2}{3} t \cdot \sqrt[2]{t} - \frac{2}{7} t^{3} \sqrt[2]{t} =$$

$$= \frac{2}{3}\, \operatorname{sen} x \cdot \sqrt[2]{\operatorname{sen} x} - \frac{2}{7} \cdot \operatorname{sen}^3 x \sqrt[2]{\operatorname{sen} x} + C =$$

$$= \operatorname{sen} x \cdot \sqrt{\operatorname{sen} x} \cdot \left(\frac{2}{3} - \frac{2}{7} \cdot \operatorname{sen}^2 x \right) + C$$

ES 25

$$\int \frac{dx}{\sqrt{(1+x^2)^3}}$$

poniamo $x = tg(t)$; $t = arctg(x)$;

$$dx = \frac{1}{\cos^2 t}\, dt$$

$1+x^2$ per la sostituzione diventa

$$1+x^2 = 1+tg^2(t) = 1+\frac{sen^2 t}{\cos^2 t} = \frac{\cos^2 t + sen^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}$$

per cui

$$\sqrt{\left(\frac{1}{\cos^2 t}\right)^3} = \sqrt{\left(\frac{1}{\cos^2 t}\right)^2 \cdot \left(\frac{1}{\cos^2 t}\right)} =$$

$$= \left(\frac{1}{\cos^2 t}\right) \cdot \sqrt{\frac{1}{\cos^2 t}} = \frac{1}{\cos^2 t} \cdot \frac{1}{\cos t}$$

per cui si ha

$$\int \frac{1}{\frac{1}{\cos^2 t} \cdot \frac{1}{\cos t}} \quad \frac{1}{\cos^2 t} \, dt =$$

$$= \int \frac{1}{\frac{1}{\cos^3 t}} \cdot \frac{1}{\cos^2 t} \, dt =$$

$$= \int \cos^{\cancel{3}} t \cdot \frac{1}{\cancel{\cos^2 t}} \, dt = \int \cos t \, dt = \operatorname{sen} t =$$

$$= \operatorname{sen}\left(\operatorname{arctg} x\right) + C$$

$$\int \frac{tg\,x}{3+\cos^2 x}\,dx \qquad (1)$$

Per sostituzione poniamo $\quad tg\,x = t \qquad (2)$

Scriviamo $\cos^2 x$ in funzione di $tg\,x$

$$tg\,x = \frac{sen\,x}{\cos x}\ ; \qquad tg^2 x = \frac{sen^2 x}{\cos^2 x}\ ;$$

$$tg^2 x = \frac{1-\cos^2 x}{\cos^2 x}$$; moltiplichiamo ambo i membri per $\cos^2 x$.

$$tg^2 x \cdot \cos^2 x = \frac{1-\cos^2 x}{\cos^2 x} \cdot \cos^2 x\ ;$$

$$tg^2 x \cdot \cos^2 x = 1-\cos^2 x\ ; \quad tg^2 x \cdot \cos^2 x - 1 + \cos^2 x = 0$$

$$\cos^2 x \left(tg^2 x + 1\right) - 1 = 0\ ; \quad \cos^2 x \left(tg^2 x + 1\right) = 1\ ;$$

(3) $\cos^2 x = \dfrac{1}{tg^2 x + 1}$. Inseriamo queste nell' integrale da calcolare

$$\int \frac{tg\,x}{3+\dfrac{1}{tg^2 x+1}}\,dx$$

visto che vogliamo applicare la sostituzione $tg\, x = t$

differenziamo $(tg\, x)' \cdot dx = dt$

$1 + tg^2 x \quad dx = dt \quad ; \quad dx = \dfrac{dt}{1+tg^2 x}$

sostituiamo $tg\, x = t$

$$\int \frac{t}{3+\frac{1}{t^2+1}} \cdot \frac{dt}{1+tg^2 x} =$$

$$= \int \frac{t}{\frac{3(t^2+1)+1}{t^2+1}} \cdot \frac{1}{1+t^2}\, dt =$$

$$= \int t \cdot \frac{\cancel{t^2+1}}{3(t^2+1)+1} \cdot \frac{1}{\cancel{1+t^2}}\, dt =$$

$$= \int \frac{t}{3t^2+3+1}\, dt = \int \frac{t}{3t^2+4}\, dt =$$

La derivata di $D(x)$ è $D'(x) = 6t$. Moltiplichiamo e dividiamo per 6.

$$= \int \frac{6}{6} \frac{t}{3t^2+4}\, dt = \frac{1}{6} \int \frac{6t}{3t^2+4}\, dt =$$

$$= \frac{1}{6} \ln\left|3t^2+4\right| = \frac{1}{6} \ln\left|3\, tg^2 x + 4\right| + C .$$

Es.

$$\int \frac{6}{\sqrt{8-3x}}\,dx$$

poniamo $\sqrt{8-3x} = t$; $8-3x = t^2$; $-3x = t^2-8$;

$3x = 8-t^2$; $3\,dx = -2t\,dt$; $dx = \frac{-2t}{3}\,dt$

$$\int \frac{6}{\sqrt{8-3x}}\,dx = \int \frac{\cancel{6}^{\,2}}{\cancel{t}} \cdot \left(-\frac{2}{\cancel{3}}\cancel{t} \cdot dt\right)$$

$$= -\int 4\,dt = -4t = -\sqrt{8-3x} + C \Rightarrow$$

$$\int \frac{6}{\sqrt{8-3x}}\,dx = -\sqrt{8-3x} + C$$

ok

ES.

$$\int \frac{1}{1+2\sqrt{x}}\,dx$$

poniamo $2\sqrt{x}=t$, $4x=t^2$, $4\,dx=2t\,dt$,

$$dx=\frac{\cancel{2}t\,dt}{2\cancel{4}}\ ;\quad dx=\frac{t\,dt}{2}.$$

$$\int \frac{1}{1+2\sqrt{x}}\,dx=\int \frac{1}{1+t}\cdot\frac{t\,dt}{2}=$$

$$=\frac{1}{2}\int\frac{t\,dt}{1+t}.$$

Poniamo $1+t=u$, $dt=du$

$$\frac{1}{2}\int\frac{u-1}{u}\,du=\frac{1}{2}\int 1-\frac{1}{u}\,du=$$

$$=\frac{1}{2}\left[\int 1\,du-\int\frac{1}{u}\,du\right]=\frac{1}{2}\left[u-\ln(u)\right]=\frac{1}{2}u-\frac{1}{2}\ln(u)$$

essendo $u=t+1$ si ottiene

$$=\frac{1}{2}(t+1)-\frac{1}{2}\ln(t+1)$$ ed essendo $t=2\sqrt{x}$ si ottiene

$$=\frac{1}{2}\left(2\sqrt{x}+1\right)-\frac{1}{2}\ln\left(2\sqrt{x}+1\right)=$$

$$= \sqrt{x} + 1 - \frac{1}{2} \ln\left(2\sqrt{x} + 1\right) + c$$

accorpiamo la costante 1 a c

$$= \sqrt{x} - \frac{1}{2} \ln\left(2\sqrt{x} + 1\right) + C.$$

OK

ES

$$\int \frac{1}{x^2+a^2}\,dx$$

mettiamo in evidenza a^2 :

$$\int \frac{1}{a^2\left(1+\frac{x^2}{a^2}\right)}\,dx = \int \frac{1}{1+\frac{x^2}{a^2}} \cdot \frac{1}{a^2}\,dx$$

poniamo la sostituzione $t = \frac{x}{a}$, $dt = \frac{1}{a}\,dx$,

$dx = a\,dt$.

Da cui

$$\int \frac{1}{1+t^2} \cdot \frac{1}{a^{\cancel{2}}}\,\cancel{a} \cdot dt = \frac{1}{a}\int \frac{1}{1+t^2}\,dt =$$

$$= \frac{1}{a}\,\mathrm{arctg}(t) = \frac{1}{a}\,\mathrm{arctg}\left(\frac{x}{a}\right)$$ OK

ES: 354

$$\int \frac{x}{\sqrt{2x+1}}\,dx$$

poniamo $\sqrt{2x+1} = t$

$2x+1=t^2$, $2x=t^2-1$, $x=\frac{t^2-1}{2}$

$dx=\frac{1}{\cancel{2}}(\cancel{2}t)\,dt = t\,dt$; $dx = t\,dt$

Da cui:

$$\int \frac{\frac{t^2-1}{2}}{\cancel{t}} \cdot \cancel{t}\,dt = \int \frac{t^2-1}{2}\,dt = \frac{1}{2}\int t^2-1\,dt =$$

$$= \frac{1}{2}\left[\int t^2\,dt - \int dt\right] = \frac{1}{2}\left[\frac{1}{3}t^3 - t\right] =$$

$$= \frac{1}{2}\left[\frac{1}{3}\left(\sqrt{2x+1}\right)^3 - \sqrt{2x+1}\right] =$$

$$= \frac{1}{2}\left[\frac{1}{3}\left(\sqrt{2x+1}\right)^2 \cdot \sqrt{2x+1} - \sqrt{2x+1}\right] =$$

$$= \frac{1}{2}\left[\frac{1}{3}(2x+1)\cdot\sqrt{2x+1} - \sqrt{2x+1}\right] =$$

$$= \frac{1}{2}\left[\sqrt{2x+1}\left(\frac{1}{3}(2x+1) - 1\right)\right] =$$

$$= \frac{1}{2}\left[\sqrt{2x+1}\left(\frac{2}{3}x + \frac{1}{3} - 1\right)\right] = \frac{1}{2}\left[\sqrt{2x+1}\left(\frac{2}{3}x + \frac{1-3}{3}\right)\right] =$$

$$= \frac{1}{2}\left[\sqrt{2x+1}\left(\frac{2}{3}x - \frac{2}{3}\right)\right] = \frac{1}{2}\left[\sqrt{2x+1}\cdot\frac{2}{3}(x-1)\right] =$$

$$= \frac{1}{2}\cdot\frac{2}{3}\left[\sqrt{2x+1}\cdot(x-1)\right] = \frac{1}{3}\cdot\sqrt{2x+1}\cdot(x-1) + C$$

ok

Es:

$$\int \frac{1}{x-\sqrt{x}}\,dx$$

poniamo $\sqrt{x}=t$, $x=t^2$, $dx=2t\,dt$

$$\int \frac{1}{t^2-t}\cdot 2t\cdot dt=\int \frac{2t}{t^2-t}\,dt=$$

$$=2\int \frac{t}{t^2-t}\,dt=2\int \frac{\cancel{t}}{\cancel{t}(t-1)}\,dt=2\int \frac{1}{t-1}\,dt=$$

$$=2\ln|t-1|=2\ln|\sqrt{x}-1|=\ln(\sqrt{x}-1)^2$$

OK

ES.:

$$\int \frac{\sqrt{x}}{x+2}\,dx$$

$$\sqrt{x} = t$$
$$x = t^2$$
$$dx = 2t\,dt$$

$$\int \frac{t}{t^2+2} \cdot 2t\,dt =$$

$$= 2\int \frac{t^2}{t^2+2}\,dt = 2\int \frac{t^2+2-2}{t^2+2}\,dt = 2\int \frac{t^2+2}{t^2+2} - \frac{2}{t^2+2}\,dt =$$

$$= 2\int 1 - \frac{2}{t^2+2}\,dt = 2\left[\int dt - \int \frac{2}{t^2+2}\,dt\right] =$$

$$= 2\left[t - 2\int \frac{1}{t^2+2}\,dt\right] = 2\left[t - 2\int \frac{1}{2\left(1+\frac{t^2}{2}\right)}\,dt\right] =$$

$$= 2\left[t - \frac{2}{2}\int \frac{1}{1+\frac{t^2}{2}}\,dt\right] = 2\left[t - \int \frac{1}{1+\left(\frac{t}{\sqrt{2}}\right)^2}\,dt\right] =$$

sapendo che $\int \frac{1}{1+x^2}\,dx = \operatorname{arctg} x$

$$= 2\left[t - \operatorname{arctg}\left(\frac{t}{\sqrt{2}}\right)\right] = 2\left[\sqrt{x} - \operatorname{arctg}\left(\frac{\sqrt{x}}{\sqrt{2}}\right)\right] =$$

$$= 2\sqrt{x} - 2\operatorname{arctg}\left(\sqrt{\frac{x}{2}}\right) + C$$

ok

ES:

$$\int \frac{\operatorname{sen} x + 3}{2 \operatorname{sen} x}\, dx \qquad R: \frac{1}{2}x + \frac{3}{2} \ln\left|tg\frac{x}{2}\right| + c$$

$$= \int \frac{\cancel{\operatorname{sen} x}}{2 \cancel{\operatorname{sen} x}}\, dx + \int \frac{3}{2 \operatorname{sen} x}\, dx = \int \frac{1}{2}\, dx + \frac{3}{2}\int \frac{1}{\operatorname{sen} x}\, dx =$$

$$= \frac{1}{2}x + \frac{3}{2}\int \frac{1}{\operatorname{sen} x}\, dx =$$

sapendo dalla tabella che $\int \frac{1}{\operatorname{sen} x}\, dx = \ln\left|tg\frac{x}{2}\right|$ si ha

$$= \frac{1}{2}x + \frac{3}{2} \cdot \ln\left|tg\frac{x}{2}\right| = \frac{1}{2}x + \ln\left|tg\frac{x}{2}\right|^{\frac{3}{2}} =$$

$$= \frac{1}{2}x + \ln\sqrt{\left|tg\frac{x}{2}\right|^3} = \frac{1}{2}x + \ln\left(\left|tg\frac{x}{2}\right| \cdot \sqrt{\left|tg\frac{x}{2}\right|}\right)$$

OK

ES:

$$\int \frac{e^x}{e^x - e^{-x}}\, dx$$

poniamo $e^x = t$

$e^x\, dx = dt$, $dx = \dfrac{dt}{e^x}$

inoltre $e^{-x} = \dfrac{1}{e^x} = \dfrac{1}{t} = t^{-1}$

$$\int \frac{\cancel{t}}{t - \frac{1}{t}} \frac{1}{\cancel{t}}\, dt = \int \frac{1}{\frac{t^2-1}{t}}\, dt = \int \frac{t}{t^2-1}\, dt =$$

$$= \int \frac{2}{2} \frac{t}{t^2-1}\, dt = \frac{1}{2} \int \frac{2t}{t^2-1}\, dt = \frac{1}{2} \operatorname{Ln}|t^2-1| =$$

$$= \frac{1}{2} \operatorname{Ln}\left|(e^x)^2 - 1\right| = \frac{1}{2} \cdot \operatorname{Ln}\left|e^{2x} - 1\right| = \operatorname{Ln}\left|e^{2x} - 1\right|^{\frac{1}{2}} =$$

$$= \operatorname{Ln} \sqrt{\left|e^{2x} - 1\right|}$$

OK

ES 387

$$\int \frac{4}{1+\cos x}\,dx$$

utilizziamo le formule parametriche

$$\operatorname{sen} x = \frac{2t}{1+t^2}\ ; \quad \cos x = \frac{1-t^2}{1+t^2}\ ; \quad \text{con } t = tg\,\frac{x}{2}\ ;$$

$$\frac{x}{2} = \operatorname{arctg} t\ ; \quad x = 2\operatorname{arctg} t\ ; \quad dx = \frac{2}{1+t^2}\,dt$$

Da cui

$$\int \frac{4}{1+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} \cdot dt$$

$$\int \frac{4}{\frac{1+t^2+1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$$

$$\int \frac{4}{\frac{2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt =$$

$$\int 4 \cdot \frac{1+t^2}{\cancel{2}} \cdot \frac{\cancel{2}}{1+t^2}\,dt = \int 4\,dt = 4\int dt = 4t =$$

$$= 4\,tg\,\frac{x}{2} + C$$

OK

ES.: 403

$$\int x \cdot \cos x \, dx \qquad \int f \cdot g' = f \cdot g - \int f' \cdot g$$

fattore finito $f(x) = x \Rightarrow f'(x) = 1$

" differenziale $g'(x) = \cos x \qquad g(x) = \operatorname{sen} x$

$$\int x \cdot \cos x \, dx = x \cdot \operatorname{sen} x - \int \operatorname{sen} x \, dx = x \cdot \operatorname{sen} x + \cos x + C$$

ok

~

ES.: 404

$$\int x \cdot \ln x \, dx =$$

$f(x) = \ln x$

$g'(x) = x$

$f'(x) = \frac{1}{x}$

$g(x) = \frac{1}{2} x^2$

$$= \ln x \cdot \frac{1}{2} x^2 - \int \frac{1}{\cancel{x}} \cdot \frac{1}{2} \cdot x^{\cancel{2}} \, dx = \frac{1}{2} x^2 \cdot \ln x - \frac{1}{2} \int x \, dx =$$

$$= \frac{1}{2} x^2 \cdot \ln x - \frac{1}{2} \cdot \frac{1}{2} x^2 = \frac{1}{2} x^2 \cdot \ln x - \frac{1}{4} x^2 = \frac{1}{4} x^2 \left(2 \ln x - 1 \right) + C$$

~

ES.: 405

$$\int x \cdot e^{-x} \, dx$$

$f(x) = x$

$g'(x) = e^{-x}$

$f'(x) = 1$

$g(x) = -\frac{1}{e^x}$

$$g(x) = \int g'(x) \, dx = \int e^{-x} \, dx = \int \frac{1}{e^x} \, dx$$

poniamo $e^x = t$, $e^x \cdot dx = dt$, $dx = \frac{dt}{e^x} = \frac{dt}{t}$

$$\int \frac{1}{t} \cdot \frac{1}{t} \, dt = \int \frac{1}{t^2} \, dt = \int t^{-2} \, dt = \frac{1}{-2+1} \cdot t^{-2+1} =$$

$$= -t^{-1} = -\frac{1}{t} = -\frac{1}{e^x}$$

Da cui $g(x) = -\frac{1}{e^x}$

Torniamo al nostro integrale

$$\int x \cdot e^{-x} \, dx = x \cdot \left(-\frac{1}{e^x}\right) - \int 1 \cdot \left(-\frac{1}{e^x}\right) dx =$$

$$= -\frac{x}{e^x} + \int \frac{1}{e^x} \, dx = -\frac{x}{e^x} + \int e^{-x} =$$ ma tale integrale lo abbiamo calcolato poco fa.

$$= -\frac{x}{e^x} + \left(-\frac{1}{e^x}\right) = -\frac{x}{e^x} - \frac{1}{e^x} = -\frac{1}{e^x}(x+1) =$$

$$= -e^{-x}(x+1) + C$$

ok

ES 407

$$\int 2x \cdot \ln x \, dx$$

$f(x) = \ln x$

$g'(x) = 2x$

$f' = \frac{1}{x}$

$g(x) = x^2$

$$g(x) = \int 2x \, dx = 2 \int x \, dx = 2 \cdot \frac{1}{2} \cdot x^2 =$$

$$= x^2$$

$$\int 2x \cdot \ln(x) \, dx = \ln x \cdot x^2 - \int \frac{1}{x} \cdot x^2 \, dx = x^2 \cdot \ln x - \int x \, dx =$$

$$= x^2 \cdot \ln(x) - \frac{1}{2} x^2 = x^2 \left(\ln(x) - \frac{1}{2} \right) + C$$

ES: 408

$f(x) = 3x$

$g'(x) = \cos x$

$g(x) = \operatorname{sen} x$

$f'(x) = 3$

$$\int 3x \cdot \cos(x) \, dx =$$

Torniamo all'integrale

$$\int 3x \cdot \cos(x) \, dx = 3x \cdot \operatorname{sen} x - \int 3 \cdot \operatorname{sen} x \, dx = 3x \cdot \operatorname{sen} x - 3 \int \operatorname{sen} x =$$

$$= 3x \cdot \operatorname{sen} x - 3(-\cos x) = 3x \cdot \operatorname{sen} x + 3\cos x = 3(x \cdot \operatorname{sen} x + \cos x) + C$$

ES: 409

$$\int x \cdot e^x \, dx$$

$f(x) = x$

$f'(x) = 1$

$g(x) = e^x$

$g'(x) = e^x$

$$\int x \cdot e^x \, dx = x \cdot e^x - \int 1 \cdot e^x \, dx = x \cdot e^x - \int e^x \, dx =$$

$$= x \cdot e^x - e^x = e^x(x-1) + C$$

ES: 4 10

$$\int \frac{\ln x}{x^2}\, dx$$

$f(x) = \frac{1}{x^2} = x^{-2}$

$f'(x) = -2 \cdot x^{-3}$

$g'(x) = \ln x$

$g(x) = x \cdot \ln x - x$

$$\int \frac{\ln x}{x^2}\, dx = x^{-2} \cdot (x \ln x - x) - \int -2x^{-3} \cdot (x \cdot \ln x - x)\, dx =$$

$$= x^{-1} \ln x - x^{-1} + \int 2x^{-3} \cdot (x \ln x - x)\, dx =$$

$$= \frac{\ln x}{x} - \frac{1}{x} + 2 \int x^{-3} \cdot (x \ln x - x)\, dx =$$

$$= \frac{\ln x}{x} - \frac{1}{x} + 2 \left[\int x^{-2} \ln x\, dx - x^{-2}\, dx \right] =$$

$$= \frac{\ln x}{x} - \frac{1}{x} + 2 \left[\int x^{-2} \ln x\, dx - \int x^{-2}\, dx \right] =$$

$$= \frac{\ln x}{x} - \frac{1}{x} + 2\left[\int x^{-2} \cdot \ln x \, dx - \frac{1}{-2+1} x^{-2+1}\right] =$$

$$= \frac{\ln x}{x} - \frac{1}{x} + 2 \int x^{-2} \cdot \ln x \, dx - \frac{2}{-1} x^{-1} =$$

$$= \frac{\ln x}{x} - \frac{1}{x} + 2 \int \frac{\ln x}{x^2} dx + 2 \cdot x^{-1} \;;$$

Riassumiamo

$$\int \frac{\ln x}{x^2} dx = \frac{\ln x}{x} - \frac{1}{x} + 2 \int \frac{\ln x}{x^2} dx + \frac{2}{x} \;;$$

$$\int \frac{\ln x}{x^2} dx - 2 \int \frac{\ln x}{x^2} dx = \frac{\ln x}{x} - \frac{1}{x} + \frac{2}{x} \;;$$

$$- \int \frac{\ln x}{x^2} dx = \frac{\ln x}{x} + \frac{1}{x} \;;$$

$$\int \frac{\ln x}{x^2} dx = - \frac{\ln x}{x} - \frac{1}{x} \;;$$

$$\int \frac{\ln x}{x^2} dx = - \frac{1}{x} \left(\ln x + 1\right) + C$$

0/1

ES 411

$$\int \operatorname{arctg} x \, dx$$

$f(x) = \operatorname{arctg}(x)$

$g'(x) = 1$

$f'(x) = \dfrac{1}{1+x^2}$

$g(x) = x$

$$\int \operatorname{arctg} x \, dx = \operatorname{arctg}(x) \cdot x - \int \frac{1}{1+x^2} \cdot x \, dx =$$

$$= \operatorname{arctg}(x) \cdot x - \int \frac{2}{2} \frac{x}{1+x^2} \, dx = \operatorname{arctg}(x) - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx =$$

$$= \operatorname{arctg}(x) - \frac{1}{2} \ln|1+x^2| + C$$

OK

ES.: 412

$$\int 4x \cdot e^{2x}\, dx =$$

$f(x) = 4x$

$g'(x) = e^{2x}$

$f'(x) = 4$

$g(x) = \dfrac{e^{2x}}{2}$ (dopo il calcolo sottostante)

$g(x) = \int g'(x)\, dx = \int e^{2x}\, dx =$ poniamo $e^x = t$, $e^x dx = dt$

$$\int t^2 \cdot \frac{1}{e^x}\, dt = \int \frac{t^2}{t}\, dt = \int t\, dt = \frac{1}{2} t^2 = \frac{1}{2} e^{2x}$$

Torniamo all'integrale

$$\int 4x \cdot e^{2x}\, dx = 4x \cdot \frac{e^{2x}}{2} - \int \cancel{4}^{2} \cdot \frac{e^{2x}}{\cancel{2}}\, dx =$$

$$= 2x \cdot e^{2x} - 2\int e^{2x}\, dx$$

Ma l'integrale $\int e^{2x}\, dx$ lo abbiamo già calcolato quando abbiamo calcolato $g(x)$. Per cui

$$\int 4x \cdot e^{2x}\, dx = 2x \cdot e^{2x} - \cancel{2} \cdot \frac{1}{\cancel{2}} e^{2x} = e^{2x}(2x-1).$$

$$\int 4x \cdot e^{2x}\, dx = e^{2x}(2x-1) + C$$

OK

ES: 460

$$\int \frac{2x+1}{x^2+x}\,dx$$

$$(x^2+x)' = 2x+1$$

$$\int \frac{2x+1}{x^2+x}\,dx = \ln|x^2+x|+c$$

~

ES: 461

$$\int \frac{4x+12}{x^2+6x}\,dx$$

$$(x^2+6x)' = 2x+6$$

$$\int \frac{2}{2}\,\frac{4x+12}{x^2+6x}\,dx = 2\int \frac{4x+12}{2x^2+12x}\,dx = 2\ln|2x^2+12x|$$

~

ES 463

$$\int \frac{x^2-1}{x^3-3x+1}\,dx$$

$$(x^3-3x+1)' = 3x^2-3 = 3(x^2-1)$$

$$\int \frac{3}{3}\,\frac{x^2-1}{x^3-3x+1}\,dx = \frac{1}{3}\int \frac{3(x^2-1)}{x^3-3x+1}\,dx = \frac{1}{3}\int \frac{3x^2-3}{x^3-3x+1}\,dx =$$

$$= \frac{1}{3}\ln|x^3-3x+1| = \ln|x^3-3x+1|^{\frac{1}{3}} = \ln\sqrt[3]{|x^3-3x+1|}+c$$

ES 477

$$\int \frac{x^2+1}{x+1}\,dx$$

$$\begin{array}{rrr|l}
x^2 & +0x & +1 & x+1 \\
\cline{4-4}
-x^2 & -x & & x-1 \\
\hline
/ & -x & +1 & \\
 & +x & +1 & \\
\cline{2-3}
 & / & +2 &
\end{array}$$

$$x^2+1=(x-1)(x+1)+2$$

$$\frac{x^2+1}{x+1}=\frac{(x-1)\cancel{(x+1)}}{\cancel{(x+1)}}+\frac{2}{x+1};$$

$$\frac{x^2+1}{x+1}=x-1+\frac{2}{x+1};$$

$$\int \frac{x^2+1}{x+1}\,dx=\int x-1+\frac{2}{x+1}\,dx=$$

$$=\int x\,dx-\int dx+2\int\frac{1}{x+1}\,dx=\frac{1}{2}x^2-x+2\ln|x+1|=$$

$$=\frac{1}{2}x^2-x+\ln(x+1)^2+C$$

Es 478

$$\int \frac{x^2+x+1}{x-4}\,dx$$

$$\begin{array}{rrr|l} x^2 & +x & +1 & x-4 \\ \cline{4-4} -x^2 & +4x & & x+5 \\ \hline / & 5x & +1 & \\ & -5x & +20 & \\ \cline{2-3} & / & 21 & \end{array}$$

$$x^2+x+1=(x-4)(x+5)+21$$

$$\frac{x^2+x+1}{x-4}=x+5+\frac{21}{x-4}$$

$$\int \frac{x^2+x+1}{x-4}\,dx=\int x+5+\frac{21}{x-4}\,dx=$$

$$=\int x\,dx+5\int dx+21\int\frac{1}{x-4}\,dx=$$

$$=\frac{1}{2}x^2+5x+21\cdot \ln|x-4|+C$$

ES: 479

$$\int \frac{x^2 - x + 3}{3 - x}\, dx$$

$$\begin{array}{rrr|l} x^2 & -x & +3 & -x+3 \\ \hline -x^2 & +3x & & -x-2 \\ \hline / & +2x & +3 & \\ & -2x & +6 & \\ \hline & / & 9 & \end{array}$$

$$x^2 - x + 3 = (-x+3)(-x-2) + 9$$

$$\frac{x^2 - x + 3}{3 - x} = (-x-2) + \frac{9}{3-x}$$

$$\int \frac{x^2 - x + 3}{3 - x}\, dx = \int -x - 2 + \frac{9}{3-x}\, dx =$$

$$= -\frac{1}{2}x^2 - 2x + 9\int \frac{1}{-x+3}\, dx =$$

$$= -\frac{1}{2}x^2 - 2x + 9\int \frac{-1}{-1} \cdot \frac{1}{-x+3}\, dx =$$

$$= -\frac{1}{2}x^2 - 2x - 9\int \frac{-1}{-x+3}\, dx = -\frac{1}{2}x^2 - 2x - 9 \cdot \ln|3-x| + C$$

OK

ES: 481

$$\int \frac{6}{x^2-9}\,dx$$

$$= 6\int \frac{1}{x^2-9}\,dx = 6\int \frac{1}{(x-3)(x+3)}\,dx$$

$$\frac{1}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3} = \frac{A(x+3)+B(x-3)}{(x-3)(x+3)} =$$

$$= \frac{Ax+3A+Bx-3B}{\sim} = \frac{x(A+B)+(3A-3B)}{\sim}$$

per l'identità si deve avere

$$\begin{cases} A+B=0 \\ 3A-3B=1 \end{cases} \quad \begin{cases} A=-B \\ -3B-3B=1 \end{cases} \quad \begin{cases} A=-B \\ -6B=1 \end{cases} \quad \begin{cases} A=+\frac{1}{6} \\ B=-\frac{1}{6} \end{cases}$$

Da cui

$$= 6\int \frac{1}{x^2-9}\,dx = 6\int \frac{\frac{1}{6}}{x-3} + \frac{-\frac{1}{6}}{x+3}\,dx =$$

$$= 6\left[\frac{1}{6}\int \frac{1}{x-3}\,dx - \frac{1}{6}\int \frac{1}{x+3}\,dx\right] =$$

$$= 6\left[\frac{1}{6}\left(\int \frac{1}{x-3}\,dx - \int \frac{1}{x+3}\,dx\right)\right]$$

$$= \ln|x-3| - \ln|x+3| = \ln\left|\frac{x-3}{x+3}\right| + C$$

$\sim$

Es: 483

$$\int \frac{1}{x^2+x}\,dx$$

$$= \int \frac{1}{x(x+1)}\,dx$$

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1)+Bx}{x(x+1)} =$$

$$= \frac{Ax+A+Bx}{x(x+1)} = \frac{x(A+B)+A}{\sim}$$

$$\begin{cases} A+B=0 \\ A=1 \end{cases} \quad \begin{cases} 1+B=0 \\ A=1 \end{cases} \quad \begin{cases} B=-1 \\ A=1 \end{cases}$$

$$\int \frac{1}{x^2+x}\,dx = \int \frac{1}{x} + \frac{-1}{x+1}\,dx =$$

$$= \int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx = \ln|x| - \ln|x+1| =$$

$$= \ln\left|\frac{x}{x+1}\right| + C$$

$\sim$

Es 490

$$\int \frac{2}{x^2-6x+9}\,dx = 2\int \frac{1}{x^2-6x+9}\,dx$$

$$x^2-6x+9=0, \quad x=\frac{6\pm\sqrt{36-36}}{2}=\frac{6}{2}=3, \quad x^2-6x+9=(x-3)^2$$

$$\frac{1}{(x-3)^2}=\frac{1}{(x-3)^2}=\left(\frac{A}{x-3}+\frac{B}{(x-3)^2}\right)=$$

$$=\left(\frac{A(x-3)+B}{(x-3)^2}\right)=\left(\frac{Ax-3A+B}{(x-3)^2}\right)=$$

$$=\left(\frac{Ax+(B-3A)}{(x-3)^2}\right)=$$

$$\begin{cases} A=0 \\ B-3A=1 \end{cases} \quad \begin{cases} A=0 \\ B=1 \end{cases}$$

Per cui

$$\int \frac{1}{x^2-6x+9}\,dx = \left[\int \frac{0}{x-3}+\frac{1}{(x-3)^2}\,dx\right]=$$

$$=\int \frac{1}{(x-3)^2}\,dx.$$ Poniamo $x-3=t$, $dx=dt$,

$$\int \frac{1}{(x-3)^2}\,dx = \text{poniamo} \quad x-3=t, \quad dx = dt$$

$$\int \frac{1}{t^2}\,dt = \int t^{-2}\,dt = \frac{1}{-2+1}\,t^{-2+1} =$$

$$= (-1)\cdot t^{-1} = -t^{-1} = -(x-3)^{-1} = \frac{-1}{x-3}$$

Riassumiamo

$$= 2\cdot\left(\frac{-1}{x-3}\right) = \frac{-2}{x-3} + c = \frac{2}{3-x} + c$$

$\sim$

Es: 493

$$\int \frac{x}{x^2-4x+4}\,dx$$

$$\int \frac{x}{(x-2)^2}\,dx$$

scomponiamo opportunamente la frazione

$$\frac{x}{(x-2)^2} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} = \frac{A(x-2)+B}{(x-2)^2} =$$

$$= \frac{Ax-2A+B}{\sim} = \frac{Ax+(B-2A)}{\sim}$$

$$\begin{cases} A=1 \\ B-2A=0 \end{cases} \quad \begin{cases} A=1 \\ B-2=0 \end{cases} \quad \begin{cases} A=1 \\ B=2 \end{cases}$$

Da cui

$$\int \frac{x}{x^2-4x+4}\,dx = \int \frac{1}{x-2}\,dx + \int \frac{2}{(x-2)^2}\,dx =$$

$$= \ln|x-2| + 2\int \frac{1}{(x-2)^2}\,dx \quad ;$$

Studiamo

$$\int \frac{1}{(x-2)^2}\,dx \qquad \text{poniamo } x-2=t,\ dx=dt$$

$$\int \frac{1}{t^2}\,dt = \int t^{-2}\,dt = \frac{1}{-1}\cdot t^{-1} = -(x-2)^{-1} = \frac{-1}{x-2}$$

Torniamo al nostro integrale

$$= \ln|x-2| + 2\left[\frac{-1}{x-2}\right] = \ln|x-2| - \frac{2}{x-2} + C$$

Esercizio:

$$\int \frac{1}{1+\sin(x)+\cos(x)}\,dx$$

risolviamo questo integrale per sostituzione utilizzando le formule parametriche di sin(x) e cos(x).

poniamo $tg\frac{x}{2} = t$

$\frac{x}{2} = arctg\, t$, $x = 2\, arctg\, t$

$$dx = 2 \cdot \frac{1}{1+t^2}\,dt$$

per le formule parametriche sappiamo che

$$\sin(x) = \frac{2t}{1+t^2} \quad e \quad \cos(x) = \frac{1-t^2}{1+t^2}$$

a questo punto possiamo scrivere nella variabile "t" l'integrale

$$\int \frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$$

$$\int \frac{1}{\frac{1+t^2+2t+1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$$

$$\int \frac{1}{\frac{2+2t}{1+t^2}} \cdot \frac{2}{1+t^2}\, dt$$

$$\int \frac{1+t^2}{2(1+t)} \cdot \frac{2}{1+t^2}\, dt$$

$$\int \frac{1}{1+t}\, dt = \ln(1+t) =$$

mai $t = \tan\frac{x}{2} \Rightarrow \ln\left(1+\tan\frac{x}{2}\right) + C$

OK

Esercizio $\int \cos^3 x \, dx$

$$\int \cos^3 x \, dx = \int \cos x \cdot \cos^2 x \, dx = \int \cos x \cdot (1 - sen^2 x) \, dx =$$

$$\int \cos x \, dx - \int \cos x \cdot sen^2 x \, dx = sen\, x + C - \int \cos x \cdot sen^2 x \, dx =$$

ma $\cos x$ è la derivata di $\sin x$ quindi è del tipo generalizzato

$$\int f'(x) \cdot f^2(x) \, dx = \frac{f^3(x)}{3}$$ da cui

$$= sen\, x + C - \frac{sen^3(x)}{3} + C = sen\, x\left(1 - \frac{\sin^2(x)}{3}\right) + C$$ OK

Esercizio $\int \sqrt{e^x - 1} \, dx$

poniamo $t = \sqrt{e^x - 1}$; $t = (e^x - 1)^{\frac{1}{2}}$

$$\frac{1}{2}(e^x - 1)^{-\frac{1}{2}} \cdot e^x \, dx = dt$$

$$\frac{1}{2} \frac{1}{(e^x - 1)^{\frac{1}{2}}} \cdot e^x \, dx = dt$$

$$dx = \frac{2(e^x - 1)^{\frac{1}{2}}}{e^x} \, dt$$

ma che $t = \sqrt{e^x - 1} \Rightarrow (e^x - 1)^{\frac{1}{2}} = t$

da $t = \sqrt{e^x - 1} \Rightarrow t^2 = e^x - 1$, $e^x = t^2 - 1$

$$dx = \frac{2 \cdot t}{t^2 - 1}$$

l'integrale nella variabile t diviene

$$\int t \cdot \frac{2t}{t^2 + 1} \, dt = \int \frac{2t^2}{t^2 + 1} \, dt =$$

$$= 2 \int \frac{t^2}{t^2 + 1} \, dt = 2 \int \frac{t^2 + 1 - 1}{t^2 + 1} \, dt =$$

$$= 2\int \frac{t^2+1}{t^2+1} - \frac{1}{t^2+1}\,dt =$$

$$= 2\left(\int 1\cdot dt - \int \frac{1}{t^2+1}\,dt\right) =$$

$$= 2\left(t - \int \frac{1}{t^2+1}\,dt\right) = 2t - 2\int \frac{1}{t^2+1}\,dt =$$

$$= 2t - 2\,\mathrm{arctg}\,t + c = 2(t - \mathrm{arctg}\,t) + c =$$

$$= 2\left(\sqrt{e^x - 1} - \mathrm{arctg}\sqrt{e^x - 1}\right) + C$$

Esercizio $\displaystyle\int \frac{x^2}{x+4}\,dx$

aggiungiamo e sottraiamo 16 al $N(x)$

$$\int \frac{x^2+16-16}{x+4}\,dx = \int \frac{x^2-16}{x+4} + \frac{16}{x+4}\,dx =$$

$$= \int \frac{(x-4)(x+4)}{x+4} + 16\int \frac{1}{x+4}\,dx =$$

$$= \int x-4 \;dx + 16\int \frac{1}{x+4}\,dx =$$

$$= \int x\,dx - 4\int dx + 16\, Ln|x+4| =$$

$$= \frac{1}{2}x^2 - 4x + 16\, Ln|x+4| + C$$ OK

Esercizio $\int x^3 \cdot \sqrt{1-x^2}\; dx$

lo svolgiamo per sostituzione ponendo $t = 1 - x^2$

$dt = -2x\, dx$, $dx = -\dfrac{dt}{2x}$

$x^2 = 1-t$, $x = \sqrt{1-t}$; $x = (1-t)^{\frac{1}{2}}$;

$x^3 = (1-t)^{\frac{3}{2}}$; $x^3 = \sqrt{(1-t)^3}$; $x^3 = (1-t)\cdot\sqrt{1-t}$

$dx = -\dfrac{dt}{2\cdot\sqrt{1-t}}$

$$\int (1-t)\cdot\sqrt{1-t}\cdot\sqrt{t}\cdot(-)\frac{dt}{2\sqrt{1-t}} =$$

$$= \int \frac{(1-t)\cdot\sqrt{t}}{2}\, dt = -\frac{1}{2}\int (1-t)\cdot\sqrt{t}\; dt =$$

$$= -\frac{1}{2}\int \sqrt{t} - t\sqrt{t}\cdot dt = -\frac{1}{2}\int \sqrt{t} - \sqrt{t^3}\; dt =$$

$$= -\frac{1}{2}\left[\int t\,dt - \int t^{\frac{3}{2}}\,dt\right] =$$

$$= -\frac{1}{2}\cdot\left[\frac{1}{2}t^2 - \frac{1}{1+\frac{3}{2}}\cdot t^{\frac{3}{2}+1}\right] =$$

$$= -\frac{1}{2}\cdot\left[\frac{t^2}{2} - \frac{1}{\frac{2+3}{2}}\, t^{\frac{3+2}{2}}\right] =$$

$$= -\frac{1}{2}\cdot\left[\frac{t^2}{2} - \frac{2}{5}\, t^{\frac{5}{2}}\right] = -\frac{1}{2}\cdot\left[\frac{t^2}{2} - \frac{2}{5}\cdot\sqrt{t^5}\right] =$$

$$= -\frac{1}{2}\cdot\left[\frac{t^2}{2} - \frac{2}{5}\cdot t^2\sqrt{t}\right] = -\frac{1}{2}t^2\cdot\left[\frac{1}{2} - \frac{2}{5}\sqrt{t}\right] =$$

$$= -\frac{1}{2}\left(1-x^2\right)^2\cdot\left[\frac{1}{2} - \frac{2\sqrt{1-x^2}}{5}\right] + C$$

OK